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Numerical techniques for flow problems with singularities

Paul A. Farrell[1], Alan F. Hegarty[2], John J. H. Miller[3], Eugene O'Riordan[4] and Grigorii I. Shishkin[5,*,†]

[1] *Department of Computer Science, Kent State University, Kent, OH 44242, U.S.A.*
[2] *Department of Mathematics & Statistics, University of Limerick, Limerick, Ireland*
[3] *Department of Mathematics, University of Dublin, Trinity College, Dublin 2, Ireland*
[4] *School of Mathematical Sciences, Dublin City University, Dublin 9, Ireland*
[5] *Institute of Mathematics & Mechanics, Ural Branch of RAS, Ekaterinburg, Russia*

SUMMARY

This paper deals with grid approximations to Prandtl's boundary value problem for boundary layer equations on a flat plate in a region including the boundary layer, but outside a neighbourhood of its leading edge. The perturbation parameter $\varepsilon = Re^{-1}$ takes any values from the half-interval $(0, 1]$; here Re is the Reynolds number. To demonstrate our numerical techniques we consider the case of the self-similar solution. By using piecewise uniform meshes, which are refined in a neighbourhood of the parabolic boundary layer, we construct a finite difference scheme that converges ε-uniformly. We present the technique of experimental substantiation of ε-uniform convergence for both the numerical solution and its normalized (scaled) difference derivatives, outside a neighbourhood of the leading edge of the plate. By numerical experiments we demonstrate the efficiency of numerical techniques based on the fitted mesh method. We discuss also the applicability of fitted operator methods for the numerical approximation of the Prandtl problem. It is shown that the use of meshes refined in the parabolic boundary layer region is necessary for achieving ε-uniform convergence. Copyright © 2003 John Wiley & Sons, Ltd.

KEY WORDS: boundary-layer equations; difference schemes; piecewise-uniform fitted mesh; fitted operator method; ε-uniform convergences

1. INTRODUCTION

Mathematical modelling of laminar flows of incompressible fluid for large Reynolds numbers Re often leads to a study of boundary value problems for boundary layer equations. Those quasilinear equations are singularly perturbed, with the perturbation parameter ε defined

* Correspondence to: G. I. Shishkin, Institute of Mathematics and Mechanics, Ural Branch of the Russian Academy of Sciences, 16 S Kovalevskaya Street, 620219 GSP-384 Ekaterinburg, Russia.
† E-mail: shishkin@imm.uran.ru

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by $\varepsilon = Re^{-1}$. The presence of parabolic boundary layers, i.e. layers described by parabolic equations, is typical for such problems [1, 2].

The difficulties arising in the numerical solution even of linear singularly perturbed equations are well known. So, the application of numerical methods developed for regular boundary value problems (see, e.g. References [3, 4]) to the problems in question yield error bounds which depend on the parameter ε. For small values of ε, the errors may be comparable to, or even much larger than the solution of the boundary value problem. This behaviour of the approximate solutions requires the development of numerical methods whose errors are independent of the parameter ε, i.e. ε-uniformly convergent methods. The presence of a non-linearity makes it considerably more difficult to construct ε-uniformly convergent numerical methods. For example, even in the case of ordinary differential quasilinear equations there do not exist fitted operator methods that converge ε-uniformly (see, e.g., References [5, 6]). This negative result has been also shown for linear problems with a parabolic boundary layer, for example, in References [7–9]. Thus, the development of ε-uniform numerical methods for resolving boundary layer equations is of considerable interest.

At present, finite difference schemes convergent ε-uniformly in the maximum norm are developed and studied for wide classes of linear singularly perturbed problems, including problems with a parabolic boundary layer (see, e.g., References [8–10]). It often occurs that the theoretical orders of ε-uniform convergence are quite low and would seem to imply that the constructed schemes will yield errors too large for practical use of these schemes. However, numerical results show that the actual convergence orders are close to those typical for regular problems (see, e.g., References [11, 12]). Thus, the experimental technique for *a posteriori* estimation of the parameters in error bounds seems to be crucial for problems with rather complicated behaviour of the solution. Note that our meaning of an *a posteriori* estimation is not related to the concept of *a posteriori* control in adaptive methods. Here, we use *a posteriori* estimation to generate an estimate of the error in a numerical solution after the computations have been completed.

It is of interest to apply the existing technique to the construction of ε-uniformly convergent schemes for boundary layer equations in that part of the boundary region where the layer is parabolic. Note that, because of the non-linearity of the boundary layer equations, the existing technique for justifying convergence and *a priori* estimates of the exact solutions do not allow us theoretically to prove ε-uniform convergence of the numerical solutions in the L_∞-norm. In this connection, we are forced to use only the alternative *a posteriori* method to study convergence, in particular, ε-uniform convergence of the numerical solutions.

In this paper, we consider grid approximations of a boundary value problem for boundary layer equations for a flat plate outside a neighbourhood of its leading edge. The boundary layer in the considered domain is parabolic. We consider the case when the solution of this classical Prandtl problem is self-similar. We construct a finite difference scheme, which is a natural development of monotone ε-uniformly convergent schemes for linear boundary value problems with a parabolic layer. For this we use standard numerical approximations on piecewise uniform meshes which are refined in the neighbourhood of the boundary layer. As is shown, the use of this fitted mesh technique that originated in Reference [8] is necessary to achieve ε-uniform convergence.

We sketch an idea of experimental *a posteriori* studying ε-uniform convergence of numerical approximations for the Prandtl problem. Note that the Prandtl problem of flow past a flat semi-infinite plate has a self-similar solution which is expressed in terms of a solution of a quasilinear third-order ODE, the so-called Blasius' equation, defined on a semi-axis. To evaluate errors in the numerical solution of the Prandtl problem, as an approximation to the self-similar solution (*reference* solution) we use a linear interpolant of the numerical solution to the Blasius equation. We study the behaviour of errors depending on both the parameter ε and the number of mesh points. This method is used to justify ε-uniform convergence of both the numerical solution and its scaled derivatives (outside a neighbourhood of the leading edge of the plate).

We emphasize the growing interest in strong numerical investigations of a boundary layer; see, for example, Reference [13]. Note that the solutions of boundary layer equations for large *Re* are close to the solution of the Navier–Stokes equations in the parabolic boundary layer region. This means that the reference solution of Prandtl's problem is the leading term in the solution of the Navier–Stokes equations at high Reynolds numbers.

We now highlight the unsolved and solved mathematical issues involved in this paper. The main aim is to develop a direct numerical method that will produce *Re*-uniformly accurate solutions to the boundary layer equations, for which flow past a flat plate is a model problem. To analyse the convergence of these numerical approximations, we need either a theoretical error bound or an exact solution. However, for the boundary layer equations, there are currently no theoretical results on the existence and uniqueness of solutions for all values of the Reynolds number. An alternative approach to analysing any proposed numerical method is required. In this paper, the self-similar solution of the Prandtl problem involves solving the Blasius problem. Hence, to generate a reference solution for the Prandtl problem, we need to generate accurate approximations to the solution and its derivatives of the Blasius problem. In Reference [14] theoretical error bounds were derived for the numerical solutions and their discrete derivatives generated from a new numerical method applied to the Blasius problem. These computed solutions can then be used to generate reference solutions of the Prandtl problem for all values of the Reynolds number. Although, we cannot produce theoretical error bounds for the numerical solutions of the direct method applied to the Prandtl problem, we can demonstrate numerically that the numerical solutions are converging independently of the Reynolds number by comparing these numerical solutions to the reference solution (generated via Blasius).

2. PROBLEM FORMULATION

Let a flat semi-infinite plate be disposed on the semi-axis $P=\{(x,y): x\geqslant 0, y=0\}$. The problem is assumed to be symmetric with respect to the plane $y=0$; we discuss the steady flow of an incompressible fluid on both sides of P, which is laminar and parallel to the plate (no separation occurs on the plate). We consider the solution of the problem on the bounded set

$$\bar{G}, \quad \text{where } G=\{(x,y): x\in(d_1,d_2],\ y\in(0,d_0)\}, \quad d_1>0 \tag{1}$$

Let $G^0=\{(x,y): x\in[d_1,d_2], y\in(0,d_0]\}$; $\bar{G}^0=\bar{G}$. Assume $S=\bar{G}\backslash G$, $S=\bigcup S_j$, $j=0,1,2$, where $S_0=\{(x,y): x\in[d_1,d_2], y=0\}$, $S_1=\{(x,y): x=d_1, \ y\in(0,d_0]\}$, $S_2=\{(x,y): x\in(d_1,d_2], y=d_0\}$, $\bar{S}_0=S_0$; $S^0=\bar{G}\backslash G^0=S_0$. On the set $\bar{G}$, it is necessary to find the solution $U(x,y)=$

$(u(x,y), v(x,y))$ of the following Prandtl problem:

$$L^1(U(x,y)) \equiv \varepsilon \frac{\partial^2}{\partial y^2} u(x,y) - u(x,y) \frac{\partial}{\partial x} u(x,y) - v(x,y) \frac{\partial}{\partial y} u(x,y) = 0 \quad (x,y) \in G \quad \text{(2a)}$$

$$L^2 U(x,y) \equiv \frac{\partial}{\partial x} u(x,y) + \frac{\partial}{\partial y} v(x,y) = 0 \quad (x,y) \in G^0 \quad \text{(2b)}$$

$$u(x,y) = \varphi(x,y) \quad (x,y) \in S \quad \text{(2c)}$$

$$v(x,y) = \psi(x,y) \quad (x,y) \in S^0 \quad \text{(2d)}$$

Here ε is the viscosity in the case when $U(x,y)$ and x,y are dimensional quantities, and $\varepsilon = Re^{-1}$ when $U(x,y)$ and x,y are dimensionless ones. The parameter ε takes arbitrary values from (0,1].

The solution of problem (2) and (1) exists and is sufficiently smooth when the functions $\varphi(x,y)$, $\psi(x,y)$ are sufficiently smooth and, moreover, the functions $\varphi(x,y)$ and $\varphi(x,y)$ together with $\psi(x,y)$ satisfy appropriate compatibility conditions, respectively, on the sets $S^* = \bar{S}_1 \cap \{S_0 \cup \bar{S}_2\}$ (i.e. at the corner points adjoining to the side $\bar{S}_1$) and $S^{0*} = \bar{S}_1 \cap S^0$ [2]. In general, the existence and uniqueness of a solution of (2) and (1) remains an open question.

We now wish to define the boundary functions $\varphi(x,y)$ and $\psi(x,y)$ more exactly.

In the quarter plane

$$\bar{\Omega}, \quad \text{where } \Omega = \{(x,y) : x, y > 0\} \quad (3)$$

let us consider the Prandtl problem whose solution is self-similar [1]:

$$\begin{aligned} L^1(U(x,y)) &= 0 \quad (x,y) \in \Omega \\ L^2 U(x,y) &= 0 \quad (x,y) \in \bar{\Omega} \backslash P \\ u(x,y) &= u_\infty, \quad x = 0, \quad y \geqslant 0 \\ U(x,y) &= (0,0) \quad (x,y) \in P \end{aligned} \quad (4)$$

where u_∞ is the velocity of free stream at infinity; $u_\infty = 1$ for the case of dimensionless variables.

Problem (4) and (3) is a subproblem of (2) and (1). Because of the special choice of the boundary functions, a self-similar solution of problem (4) and (3) exists [1].

The self-similar solution of problem (4) and (3) can be written in terms of some function $f(\eta)$ and its derivative

$$u(x,y) = u_\infty f'(\eta), \quad v(x,y) = \varepsilon^{1/2}(2^{-1} u_\infty x^{-1})^{1/2}(\eta f'(\eta) - f(\eta)) \quad (5)$$

where $\eta = \varepsilon^{-1/2}(2^{-1} u_\infty x^{-1})^{1/2} y$. The function $f(\eta)$ is the solution of the Blasius problem

$$\begin{aligned} L(f(\eta)) &\equiv f'''(\eta) + f(\eta) f''(\eta) = 0, \quad \eta \in (0, \infty) \\ f(0) &= f'(0) = 0, \quad \lim_{\eta \to \infty} f'(\eta) = 1 \end{aligned} \quad (6)$$

In the sequel, we call the numerical approximations to the velocity components (u, v) from (5) the *reference* solutions for the Prandtl problem.

The functions $\varphi(x,y), \psi(x,y)$ in (2) are defined by[‡]

$$\varphi(x,y) = u_{(5)}(x,y) \quad (x,y)\in S, \qquad \psi(x,y) = v_{(5)}(x,y) \quad (x,y)\in S^0 \tag{7}$$

Note that $\varphi(x,y)=\varphi(x,y;\varepsilon)=0, \psi(x,y)=\psi(x,y;\varepsilon)=0, (x,y)\in S^0$. Since we are not including the leading edge, the techniques in Reference [2] are applicable for the proof of the existence and uniqueness of a solution of (2), (7) and (1).

As $\varepsilon\to 0$, the solution has a parabolic boundary layer in a neighbourhood of the set S^0.

To solve problem (2), (7) and (1) numerically, we will construct a finite difference scheme which generates ε-uniformly convergent approximations.

3. DIFFERENCE SCHEME FOR PROBLEM (2), (7) AND (1)

Assume that we know the 'coefficients' multiplying the derivatives $(\partial/\partial x)u(x,y)$ and $(\partial/\partial y)$ $u(x,y)$ in the operator $L^1_{(2)}$; let these be some functions $u_0(x,y)$ and $v_0(x,y)$. In this case the transport equation takes the form

$$Lu(x,y) \equiv \left\{\varepsilon\frac{\partial^2}{\partial y^2} - u_0(x,y)\frac{\partial}{\partial x} - v_0(x,y)\frac{\partial}{\partial y}\right\} u(x,y) = 0 \quad (x,y)\in G \tag{8}$$

The function $u_0(x,y)$ outside an $m\varepsilon$-neighbourhood of S^0 satisfies the condition [1, Chapter 7][§]

$$u_0(x,y)\geqslant m_0, \ (x,y)\in\bar{G} \quad \text{and} \quad r((x,y),S^0)\geqslant m\varepsilon^{1/2} \tag{9a}$$

and also

$$u_0(x,y)>0 \quad (x,y)\in\bar{G}, \ \ y>0 \tag{9b}$$

where $r((x,y),S^0)$ is the distance from the point (x,y) to the set S^0. By virtue of condition (9b) the operator $L_{(8)}$ is monotone [4] (i.e. a comparison principle is applicable).

For the function $v_0(x,y)$ the following estimate [1] is valid:

$$0\leqslant v_0(x,y)\leqslant M\varepsilon^{1/2}, \quad (x,y)\in\bar{G} \tag{9c}$$

This means that the product $\varepsilon^{-1/2}v_0(x,y)$ (i.e. the normalized component) is of order $O(1)$, that is, bounded ε-uniformly. Thus, by virtue of (9a)–(9c), the singular part of the solution of Equation (8) behaves similarly to the singular part for the singularly perturbed heat equation

$$Lu(x,y) \equiv \left\{\varepsilon\frac{\partial^2}{\partial y^2} - \frac{\partial}{\partial x}\right\} u(x,y) = 0 \tag{10}$$

In the case of a boundary value problem for the singularly perturbed equation (10), difference schemes on special piecewise uniform meshes are well known (see, e.g.,

‡ Here and in what follows, the notation $w_{(j.k)}$ indicates that w is first defined in equation $(j.k)$.

§ Throughout this paper, we denote by M (or m) sufficiently large (small) positive constants which are independent of the parameter ε and of the discretization parameters.

References [8, 9]). We now use such meshes in the construction of ε-uniform schemes for problem (2), (7) and (1).

To solve the boundary value problem (2), (7) and (1) numerically, we use a classical finite difference schemes. At first we introduce the rectangular grid on the set $\bar{G}$:

$$\bar{G}_h = \bar{\omega}_1 \times \bar{\omega}_2 \tag{11}$$

where $\bar{\omega}_1$ and $\bar{\omega}_2$ are meshes on the segments $[d_1, d_2]$ and $[0, d_0]$, respectively; $\bar{\omega}_1 = \{x^i,\ i = 0, \ldots, N_1, x^0 = d_1, x^{N_1} = d_2\}$, $\bar{\omega}_2 = \{y^j,\ j = 0, \ldots, N_2, y^0 = 0, y^{N_2} = d_0\}$; $N_1 + 1$ and $N_2 + 1$ are the number of nodes in the meshes $\bar{\omega}_1$ and $\bar{\omega}_2$. Define $h_1^i = x^{i+1} - x^i$, $x^i, x^{i+1} \in \bar{\omega}_1$, $h_2^j = y^{j+1} - y^j$, $y^j, y^{j+1} \in \bar{\omega}_2$, $h_1 = \max_i h_1^i$, $h_2 = \max_j h_2^j$, $h = \max[h_1, h_2]$. We assume that $h \leqslant MN^{-1}$, where $N = \min[N_1, N_2]$.

We approximate the boundary value problem by the difference scheme

$$\begin{aligned}
\Lambda^1(U^h(x,y)) &\equiv \varepsilon\delta_{\bar{y}\hat{y}} u^h(x,y) - u^h(x,y)\delta_{\bar{x}} u^h(x,y) \\
&\quad - v^h(x,y)\delta_{\bar{y}} u^h(x,y) = 0 \quad (x,y) \in G_h && \text{(12a)}\\
\Lambda_1^2 U^h(x,y) &\equiv \delta_{\bar{x}} u^h(x,y) + \delta_{\bar{y}} v^h(x,y) = 0 \quad (x,y) \in G_h^0,\ x > d_1 \\
\Lambda_2^2 U^h(x,y) &\equiv \delta_x u^h(x,y) + \delta_{\bar{y}} v^h(x,y) = 0 \quad (x,y) \in S_{1h} && \text{(12b)}\\
u^h(x,y) &= \varphi(x,y) \quad (x,y) \in S_h && \text{(12c)}\\
v^h(x,y) &= \psi(x,y) \quad (x,y) \in S_h^0 && \text{(12d)}
\end{aligned}$$

Here $\delta_{\bar{y}\hat{y}} z(x,y)$ and $\delta_x z(x,y), \ldots, \delta_{\bar{y}} z(x,y)$ are the second and first difference derivatives (the bar denotes the backward difference): $\delta_{\bar{y}\hat{y}} z(x,y) = 2(h_2^{j-1} + h_2^j)^{-1}(\delta_y z(x,y) - \delta_{\bar{y}} z(x,y))$, $\delta_x z(x,y) = (h_1^i)^{-1}(z(x^{i+1},y) - z(x,y)), \ldots, \delta_{\bar{y}} z(x,y) = (h_2^{j-1})^{-1}(z(x,y) - z(x,y^{j-1}))$, $(x,y) = (x^i, y^j)$.

If the 'coefficients' multiplying the differences $\delta_{\bar{x}}$ and $\delta_{\bar{y}}$ in the operator Λ^1 are known (let these be the functions $u_0^h(x,y)$ and $v_0^h(x,y)$) and satisfy the condition $u_0^h(x,y), v_0^h(x,y) \geqslant 0$ for $(x,y) \in \bar{G}_h$, the operator Λ^1 is monotone [4].

Let us introduce a piecewise uniform mesh refined in a neighbourhood of the set S^0. On the set $\bar{G}$, we consider the grid

$$\bar{G}_h^* = \bar{\omega}_1 \times \bar{\omega}_2^* \tag{13}$$

where $\bar{\omega}_1$ is a uniform mesh on $[d_1, d_2]$, $\bar{\omega}_2^* = \bar{\omega}_2^*(\sigma)$ is a special piecewise uniform mesh depending on the parameter σ and on the value N_2. The mesh $\bar{\omega}_2^*$ is constructed as follows. We divide the segment $[0, d_0]$ in two parts $[0, \sigma]$ and $[\sigma, d_0]$. The stepsize of the mesh $\bar{\omega}_2^*$ is constant on the segments $[0, \sigma]$ and $[\sigma, d_0]$, and equal to $h_2^{(1)} = 2\sigma N_2^{-1}$ and $h_2^{(2)} = 2(d_0 - \sigma)N_2^{-1}$, respectively. The value of σ is defined by

$$\sigma = \min[2^{-1} d_0, m\varepsilon^{1/2} \ln N_2]$$

where m is an arbitrary positive number.

In the case of the boundary value problem (2), (7) and (1), it is required to study whether the solutions of the difference scheme (12) and (13) converge to the exact solution. The solution of problem (2) and (7) on the set $\bar{G}_{(1)}$ is sufficiently smooth for fixed values of the parameter ε, but its y-derivatives grow unboundedly in a neighbourhood of the boundary layer as $\varepsilon \to 0$. The difference scheme (12) and (13) approximates problem (2), (7) and (1) on its solution with the first-order of accuracy in x and y. In that case when the functions $u^h(x, y)$ and $v^h(x, y)$ considered as the coefficients multiplying the derivatives $\delta_{\bar{x}} u^h(x, y)$ and $\delta_{\bar{y}} u^h(x, y)$ are non-negative, the difference scheme (12) and (13) is monotone.

Note that the main difference between the scheme suggested in the paper and standard well-known schemes consists in using meshes whose stepsize, transversal to the boundary layer, in a neighbourhood of the layer is small and is appropriate for the value of the parameter ε. It is obvious from the structure of the mesh (13) that its stepsize in the y direction changes abruptly at the transition point $y = \sigma$ when ε is small. This abrupt change of mesh size, generally speaking, can lead to a loss in well conditioning of a scheme. Note that this question requires a further theoretical study. Nevertheless, no loss in conditioning as compared to regular problems was revealed in numerical experiments for the values of ε and N within the broad diapasons for reaction–diffusion and convection–diffusion problems (see, e.g. results of a series of numerical experiments in References [12, 15, 16]). It is worth noting that the abrupt change in mesh size has no adverse effect on the stability of the numerical scheme (see Reference [16]). No stability difficulties associated with the use of piecewise-uniform fitted meshes were encountered in these and other numerical studies.

We mention certain difficulties that arise in studying convergence properties. In the case of ε-uniformly convergent difference schemes for linear problems, methods are well developed to determine numerically the parameters in the error bounds (orders of convergence and error constants for fixed values of ε and ε-uniformly), see, e.g. Reference [12], where ε-uniform convergence is known in advance from theoretical studies. Formally these methods are inapplicable for problem (2), (7) and (1) because the ε-uniform convergence of scheme (12) and (13) has not been established. Nevertheless, the results of such investigations of error bounds seem to be interesting for practical use.

The pointwise comparison of the exact solutions of problem (2), (7) and (1) with the solutions of difference scheme (12) and (13) gives us more detailed knowledge about the behaviour of the error bounds. To find the exact solutions of Prandtl's problem, we shall use the Blasius solution of problem (6). Note that the numerical solution of the Blasius problem yields its own additional errors. As for scheme (12) and (13), it is of great interest to study errors for computation of which we use the 'exact' solutions of the Prandtl problem obtained on the basis of the discrete solutions of Blasius' problem.

Note that the difference scheme (12) and (13) is non-linear. To find an approximate solution of this scheme, we must construct a proper iterative method.

4. ITERATIVE DIFFERENCE SCHEME FOR THE PRANDTL PROBLEM

Note that (2a) is a parabolic equation in which the variable x plays the role of time. The problem (12) and (11) is solved on levels with respect to the variable $x^i \in \bar{\omega}_1$. To find the discrete solution at the level $x^{i_0} > d_1$, we use an iterative method.

In order to define the iterative difference scheme we must specify the boundary function $\varphi(x, y)$, $(x, y) \in S_h$ ($\psi(x, y) = 0$, $(x, y) \in S_h^0$). The function $\varphi(x, y)$ has no analytical representation. Instead of the function $\varphi(x, y)$, we use a function $\varphi^h(x, y)$ which can be found by using the grid solution of the Blasius' problem.

Let us describe an iterative process used in the computation of the solution at the level x^{i_0+1} for $x^{i_0} > d_1$. Assume that the solution of the discrete problem (or its approximation) is known for $x = x^{i_0}$. The function $U^h(x, y)$ for $x = x^{i_0+1}$, $y \in \bar{\omega}_2$ is the solution of the non-linear system of algebraic equations. To compute a new iteration for the component $u^h_{k+1}(x, y)$, $x = x^{i_0+1}$, we use (12a) in which we replace the coefficients multiplying the derivatives $\delta_{\bar{x}}\, u^h_{k+1}$ and $\delta_{\bar{y}}\, u^h_{k+1}$ by the known components u^h_k and v^h_k from the previous iteration. The component $v^h_{k+1}(x, y)$, $x = x^{i_0+1}$ is computed from (12b) by using the known component u^h_{k+1}. We continue these iterations until the difference between the functions $u^h_k(x, y)$, $\varepsilon^{-1/2} v^h_k(x, y)$ for $x = x^{i_0+1}$, $y \in \bar{\omega}_2$ at the neighbouring iterations becomes less than some prescribed sufficiently small value $\delta > 0$, which defines the required accuracy of the iterative solution. As an initial guess, namely, for the function $U^h_0(x, y)$, $x = x^{i_0+1}$, we use the known solution at the level $x = x^{i_0}$.

For $x = x^{i_0} = x^0 = d^1$, to compute the grid solution at $x = x^{i_0+1}$ we use the above-described iteration process in which we choose, as an initial guess $U^h_0(x, y)$, $x = x^{i_0+1}$, the function $U^h_0(x, y) = (u^h_0(x, y) = \varphi^h(x, y), v^h_0(x, y) = \psi(x, y) = 0)$, $x = x^1$, $y \in \bar{\omega}_2$.

The function $u^h(x, y)$ at the level $x = x^0 = d^1$ is known according to the problem formulation; the function $v^h(x, y)$ is computed from (12b).

Thus, we obtain the following final iterative difference scheme:

$$\begin{aligned}
\Lambda^1(u^h_k(x, y); u^h_{k-1}(x, y), v^h_{k-1}(x, y)) &\equiv \varepsilon\delta_{\bar{y}\hat{y}} u^h_k(x, y) - u^h_{k-1}(x, y)\delta_{\bar{x}}\, u^h_k(x, y) \\
&\quad - v^h_{k-1}(x, y)\delta_{\bar{y}}\, u^h_k(x, y) = 0, \quad y \in \omega_2
\end{aligned}$$

$$\begin{aligned}
\Lambda^2_1(v^h_k(x, y); u^h_k(x, y), u^h_{K(x^{i-1})}(x^{i-1}, y)) &\equiv (x^i - x^{i-1})^{-1}[u^h_k(x, y) - u^h_{K(x^{i-1})}(x^{i-1}, y)] \\
&\quad + \delta_{\bar{y}} v^h_k(x, y) = 0, \quad y \in \bar{\omega}_2, \ \ y \neq 0
\end{aligned}$$

$$\begin{gathered}
u^h_k(x, y) = \varphi^h(x, y), \quad y = 0, d_0; \qquad v^h_k(x, y) = 0, \quad y = 0 \\
u^h_0(x, y) = \begin{cases} u^h_{K(x^{i-1})}(x^{i-1}, y), & x^i \geqslant x^2 \\ \varphi^h(x = d_1, y), & x^i = x^1, \ \ y \in \omega_2 \end{cases} \\
v^h_0(x, y) = \begin{cases} v^h_{K(x^{i-1})}(x^{i-1}, y), & x^i \geqslant x^2 \\ 0, & x^i = x^1, \ \ y \in \bar{\omega}_2, \ \ y \neq 0 \end{cases} \\
\max_{y \in \bar{\omega}_2} |u^h_K(x, y) - u^h_{K-1}(x, y)|, \ \ \varepsilon^{-1/2} \max_{y \in \bar{\omega}_2} |v^h_K(x, y) - v^h_{K-1}(x, y)| \leqslant \delta \\
\max_{k < K} \left[\max_{y \in \bar{\omega}_2} |u^h_k(x, y) - u^h_{k-1}(x, y)|, \ \ \varepsilon^{-1/2} \max_{y \in \bar{\omega}_2} |v^h_k(x, y) - v^h_{k-1}(x, y)| \right] > \delta \\
\text{for } x = x^i, \quad i = 1, \ldots, N_1, \quad k = 1, \ldots, K, \quad K = K(x^i) \\
\begin{aligned}
\Lambda^2_2(v^h(x, y); u^h_{K(x^1)}(x^1, y)) &\equiv (x^1 - x^0)^{-1}[u^h_{K(x^1)}(x, y) - \varphi^h(x, y)] \\
&\quad + \delta_{\bar{y}} v^h(x, y) = 0, \quad y \in \bar{\omega}_2, \ \ y \neq 0 \\
&\quad \text{for } x = x^0 = d_1
\end{aligned}
\end{gathered} \tag{14}$$

Scheme (14) and (13) permits us to compute the function $U^h(x,y)=(u^h(x,y),\ v^h(x,y))$, $(x,y)\in\bar{G}_h$, namely, the components $u^h_{K(x^i)}(x,y)$, $v^h_{K(x^i)}(x,y)$ for $x^i\geqslant x^1$, $y\in\bar{\omega}_2$ and the function $v^h(x,y)$ for $x^i=x^0=d_1$, $y\in\bar{\omega}_2$. We call the function $U^h(x,y)$, $(x,y)\in\bar{G}_h$ satisfying (14) the solution of the iterative difference scheme (14) and (13).

5. APPROXIMATION OF THE SELF-SIMILAR SOLUTION TO THE PRANDTL PROBLEM BY USING THE BLASIUS' EQUATION

In the case of scheme (14) and (13), to analyse the approximation error for the solutions of problem (2), (7) and (1) and their derivatives, we use the self-similar solution (5) defined by the solution of the Blasius' problem (6).

For the boundary value problem (6) we must construct a finite difference scheme that allows us to approximate both the Blasius' solution and its derivatives on the semi-axis $\eta\geqslant 0$. It is required to find 'constructive' difference schemes, i.e. difference schemes on meshes with a finite number of nodes.

We approximate problem (6) by the following differential problem on a finite interval. Let $f_\star(\eta)$, $\eta\in[0,T]$, where the length T of the interval is sufficiently large, be the solution of the boundary value problem

$$\begin{aligned} L(f_\star(\eta))\equiv f_\star'''(\eta)+f_\star(\eta)f_\star''(\eta)=0,\quad \eta\in(0,T)\\ f_\star(0)=f_\star'(0)=0,\quad f_\star'(T)=1 \end{aligned} \tag{15a}$$

We complete a definition of the function $f_\star(\eta)$ on the infinite interval (T,∞) by setting

$$f_\star(\eta)=f_\star(T)+(\eta-T)\quad \text{for all } \eta>T \tag{15b}$$

The continuous problem (15a) and (15b) is approximated by a discrete problem. For this we introduce a uniform mesh on the interval $[0,T]$ as follows:

$$\bar{\omega}_0=\{\eta^i=ih, i=0,1,\dots,N;\ \eta^0=0,\ \eta^N=T\} \tag{16}$$

with stepsize $h=TN^{-1}$, where $N+1$ is the number of nodes in the mesh $\bar{\omega}_0$. Assume $T=\ln N$. On the mesh $\bar{\omega}_0$, we approximate problem (15a) by the grid problem

$$\begin{aligned} \Lambda(f^h(\eta))\equiv\delta_{\eta\bar{\eta}\bar{\eta}}f^h(\eta)+f^h(\eta)\delta_{\eta\bar{\eta}}f^h(\eta)=0,\quad \eta\in\bar{\omega}_0,\ \ \eta\neq\eta^0,\eta^1,\eta^N\\ f^h(0)=\delta_\eta f^h(0)=0,\quad \delta_{\bar{\eta}}f^h(T)=1 \end{aligned} \tag{17a}$$

Here $\delta_{\eta\bar{\eta}}z(\eta)$ and $\delta_{\eta\bar{\eta}\bar{\eta}}z(\eta)$ are the second (centred) and third difference derivatives:

$$\delta_{\eta\bar{\eta}}z(\eta)=h^{-1}(\delta_\eta z(\eta)-\delta_{\bar{\eta}}z(\eta)),\quad \delta_{\eta\bar{\eta}\bar{\eta}}z(\eta)=h^{-1}(\delta_{\eta\bar{\eta}}z(\eta)-\delta_{\eta\bar{\eta}}z(\eta^{i-1})),\quad \eta=\eta^i$$

The function $f^h(\eta)$ on the interval (T,∞) is defined by

$$f^h(\eta)=f^h(T)+(\eta-T),\quad \eta\in(T,\infty) \tag{17b}$$

Equations (17a) and (17b) allows us to find the function $f^h(\eta)$ for $\eta\in\bar{\omega}_0$ and $\eta\in(T,\infty)$. To determine the components of the solution and their derivatives for the Prandtl problem, we need derivatives of the function $f^h(\eta)$. Let $\delta^k_\eta f^h(\eta)=\delta_\eta(\delta^{k-1}_\eta f^h(\eta))$, $\eta\in\bar{\omega}_0$, $\eta\leqslant\eta^{N-k}$,

$k \geqslant 1$, be the kth difference derivatives of $f^h(\eta)$ on $\bar{\omega}_0$. Assume $\delta^k_\eta f^h(\eta)=1$ for $k=1$, $\eta=\eta^N$ and $\delta^k_\eta f^h(\eta)=0$ for $k\geqslant 2$, $\eta\in\bar{\omega}_0$, $\eta^{N-k+1}\leqslant\eta\leqslant\eta^N$. By $\bar{f}^{h(k)}(\eta)$, $\eta\in[0,T]$, we denote the linear interpolant constructed from the values of the functions $\delta^k_\eta f^h(\eta)$, $\eta\in\bar{\omega}_0$, $k\geqslant 0$; $\delta^0_\eta f^h(\eta)=f^h(\eta)$. The function $\bar{f}^{h(k)}(\eta)$ is extended to the interval (T,∞) by the definitions: $\bar{f}^{h(k)}(\eta)=f^h(\eta)$ for $k=0$, $\bar{f}^{h(k)}(\eta)=1$ for $k=1$, $\bar{f}^{h(k)}(\eta)=0$ for $k\geqslant 2$, $\eta\in(T,\infty)$. We shall call the function $\bar{f}^h(\eta)=\bar{f}^{h(k=0)}(\eta)$, $\eta\in[0,\infty)$, defined in such a way, the solution of problem (17a), (17b) and (16) and the functions $\bar{f}^{h(k)}(\eta)$, $k\geqslant 1$, the derivatives (of order k) from the solution of problem (17a), (17b) and (16).

Problem (17a), (17b) and (16) is non-linear. Let us describe an iterative difference scheme for the approximate solution of problem (17a), (17b) and (16).

On the mesh $\bar{\omega}_{0(16)}$, we find the function $f^h_R(\eta)$ by solving successively the problems

$$\begin{gathered}\Lambda(f^h_r(\eta), f^h_{r-1}(\eta))\equiv\delta_{\eta\bar{\eta}\bar{\eta}} f^h_r(\eta)+f^h_{r-1}(\eta)\delta_{\eta\bar{\eta}} f^h_r(\eta)=0, \quad \eta\in\bar{\omega}_0,\ \eta\neq\eta^0,\eta^1,\eta^N \\ f^h_r(0)=\delta_\eta f^h_r(0)=0, \quad \delta_{\bar{\eta}} f^h_r(T)=1, \qquad r=1,\dots,R\end{gathered} \tag{18a}$$

where $f^h_0(\eta)=\eta$, $\eta\in\bar{\omega}_0$, R is a sufficiently large given number. For $\eta\in(T,\infty)$ we define the function $f^h_R(\eta)$ by setting

$$f^h_R(\eta)=f^h_R(T)+(\eta-T), \quad \eta\in(T,\infty) \tag{18b}$$

Problem (18a) and (16) is linear with respect to the function $f^h_r(\eta)$, $\eta\in\bar{\omega}$.

From the values of the function $f^h_R(\eta)$, similarly to the function $\bar{f}^{h(k)}(\eta)$, $\eta\in[0,\infty)$, we construct the function $\bar{f}^{h(k)}_*(\eta)=\bar{f}^{h(k)}_R(\eta)$, $\eta\in[0,\infty)$, $k\geqslant 0$. We call the function $\bar{f}^h_*(\eta)=\bar{f}^{h(k)}_*(\eta)$, $\eta\in[0,\infty)$ for $k=0$ the solution of problem (18a), (18b) and (16), and the functions $\bar{f}^{h(k)}_*(\eta)$, $k\geqslant 1$ the derivatives of the problem solution. Note that the derivatives of the function $\bar{f}^{h(k)}_*(\eta)$ have a discontinuity of the first kind at $\eta=\eta^{N-k}$, $k\geqslant 2$.

For

$$T=T(N)=M_1\ln N, \quad R=R(N)=M_2\ln N \tag{19}$$

where M_1, M_2 are sufficiently large numbers, the solution of problem (18a), (18b), (16) and (19) together with its derivatives up to order K (where K is fixed) converges, as $N\to\infty$, to the solution of problem (6) and to the corresponding derivatives.

Note that the differential equation in (15a) is a reaction–diffusion equation with respect to the function $\tilde{f}(\eta)=f'_*(\eta)$, moreover, the problem (15a) is monotone with respect to the function $\tilde{f}(\eta)$ (satisfies the maximum principle).

When the function $f^h(\eta)$ is non-negative, the equation (17a) is monotone with respect to the function $\tilde{f}^h(\eta)=\delta_{\bar{\eta}} f^h(\eta)$. The consideration of linear analogues of this equation shows that, in order to achieve convergence of the iterates as $N\to\infty$, it suffices that the value T and the number R of iterations satisfy condition (19). For these linear analogues we obtain first (up to a logarithmic factor) order accuracy. The parameter-uniform accuracy and stability issues associated with scheme (18a), (18b), (16) and (19) are discussed also in Reference [16, Chapters 10 and 11].

The theoretical and numerical analyses given in Reference [14] result in the estimate:

$$
\begin{aligned}
&|f'(\eta) - \bar{f}_*^{h(1)}(\eta)|, \quad |\eta f'(\eta) - f(\eta) - (\eta \bar{f}_*^{h(1)}(\eta) - \bar{f}_*^{h}(\eta))| \\
&|\eta^k (f''(\eta) - \bar{f}_*^{h(2)}(\eta))| \leqslant M N^{-v}, \quad \eta \in [0,\infty), \ \ k = 0, 1, 2
\end{aligned}
\tag{20}
$$

where v is some number ($0 < v < 1$). It follows from estimates (20) that the difference scheme (18a), (18b), (16) and (19) in the case of Prandtl's problem (2), (7) and (1) allows us to find the normalized components with the normalized (i.e. ε-uniformly bounded) derivatives, namely, $u(x,y)$, $\varepsilon^{-1/2}v(x,y)$, $(\partial/\partial x)u(x,y)$, $\varepsilon^{1/2}(\partial/\partial y)u(x,y)$, $\varepsilon^{-1/2}(\partial/\partial x)v(x,y)$, $(\partial/\partial y)v(x,y)$, $(x,y) \in \bar{G}_{(1)}$, with guaranteed (controllable) ε-uniform accuracy (see Reference [16, Chapter 11] for numerical results). Note that, by virtue of (20), the error estimate for the *reference* solution obtained in this way is independent of the parameter ε and determined only by the value of N, that is, the number of mesh intervals into which we divided the interval $[0,T]$. Because of (19), the stepsize of mesh (16) is defined by

$$
h = TN^{-1} = M_1 N^{-1} \ln N \tag{21}
$$

Thus, in the case of Prandtl's problem (2), (7) and (1) the components of its solution and the partial derivatives with respect to x and y, which are determined via the solutions of scheme (18a), (18b), (16) and (19) for Blasius' problem (6), permit us to form the boundary conditions (with controllable ε-uniform accuracy) in the iterative difference scheme (14) and (13). Besides, the solutions of scheme (18a), (18b), (16) and (19) allow us to analyse the ε-uniform convergence of special difference schemes, in particular, of schemes (14), (13) and (12), (13).

Note that this numerical method for Blasius' problem generates global approximations (valid for all $\eta \in [0,\infty)$) to the solution and its derivatives, whose accuracy is determined solely by N (the number of nodes in the interval $[0, M_1 \ln N]$). The efficiency of such a method can be contrasted with the infinite-series representation for the 'semi-analytic' solution to the Blasius' problem given in Reference [17], for which the accuracy of the truncated (with specific non-evident choices of two auxiliary parameters) series is unknown for all $\eta \in [0,\infty)$.

By numerical experiments, implemented according to the above techniques, in Reference [16, Chapter 12] we show the ε-uniform convergence of schemes (14), (13) and (12), (13) of the direct method; also therein we find the convergence orders for the numerical approximations to the solutions and derivatives for Prandtl's problem (2), (7) and (1). In Section 7, for the convenience of the reader, we repeat some minimal numerical results from [16] confirming the efficiency of the numerical method based on Blasius' approach, thus making our exposition self-contained and complete.

6. ON FITTED OPERATOR SCHEMES FOR THE PRANDTL PROBLEM

As was shown in References [8, 18] (see also References [7, 9]) for a singularly perturbed parabolic equation with parabolic boundary layers, there do not exist fitted operator schemes on uniform meshes that converge ε-uniformly. Note that the coefficients in the terms with first-order derivatives in time and second-order derivatives in space do not vanish in the equations considered in References [8, 18]. Note also that, in the Prandtl problem, the coefficient multiplying the first derivative with respect to the variable x, which plays the role of the time variable, vanishes on the domain boundary for $y=0$. Unlike the problem studied in Reference [8], where the boundary conditions do not obey any restriction, besides the requirement of sufficient smoothness, problem (2), (7) and (1) is essentially simpler. The data of this problem, i.e. the zero right-hand sides of equations (2a), (2b) and the boundary conditions (2c) and (2d), and therefore the solution itself are defined only by the one parameter u_∞. We are interested whether or not one variant of a fitted operator method, in which the fitting coefficients (depending on ε) are independent of the value u_∞, is applicable to construct ε-uniformly convergent schemes.

In Reference [19] an ε-uniform fitted operator method was constructed for a linear parabolic equation with a discontinuous initial condition in the presence of a parabolic (transient) layer. Such a fitted operator scheme was successfully constructed because all of the singular components of the solution (their main parts) are defined, up to some multiplier, by just one function. In view of the simple (depending on the one parameter u_∞) representation of the solution for the Prandtl problem, it is not obvious that for this problem there are no fitted schemes on uniform meshes which converge ε-uniformly.

We will try to construct a fitted operator scheme starting from (12a) under the assumption that the function $v^h(x,y)$ is known, where $v^h(x,y)=v(x,y)$. Let us consider a fitted operator scheme of the form

$$\begin{aligned}\Lambda^{1*}(u^h(x,y)) \equiv \varepsilon\gamma_{(2)}\delta_{y\bar{y}}\,u^h(x,y) - u^h(x,y)\delta_{\bar{x}}u^h(x,y)&\\ - \gamma_{(1)}v(x,y)\delta_{\bar{y}}\,u^h(x,y) = 0 \quad (x,y)\in G_h& \qquad (22a)\\ u^h(x,y)=\varphi(x,y) \quad (x,y)\in S_h&\end{aligned}$$

where

$$\bar{G}_h \qquad (23)$$

is a uniform rectangular grid, with steps h_1 and h_2 in x and y, respectively; the parameters

$$\gamma_{(i)}=\gamma_{(i)}(x,y;\varepsilon,h_1,h_2), \quad i=1,2 \qquad (22b)$$

are fitting coefficients.

The difference scheme (22a), (22b) and (23) is a fitted scheme for the following boundary value problem:

$$\begin{aligned}L^{1*}(u(x,y)) \equiv \varepsilon\frac{\partial^2}{\partial y^2}u(x,y) - u(x,y)\frac{\partial}{\partial x}u(x,y) - v(x,y)\frac{\partial}{\partial y}u(x,y) = 0 \quad (x,y)\in G&\\ u(x,y)=\varphi(x,y) \quad (x,y)\in S&\end{aligned} \qquad (24)$$

where the function $v(x,y)=v_{(5)}(x,y)$ is considered to be known.

The derivatives of the function $u(x,y)$ can be represented as follows:

$$\begin{aligned}&\frac{\partial}{\partial x}u(x,y) = -2^{-1}u_\infty x^{-1}f''(\eta)\eta, \quad \frac{\partial^2}{\partial x^2}u(x,y) = 4^{-1}u_\infty x^{-2}[f'''(\eta)\eta^2 + 3f''(\eta)\eta]\\ &\frac{\partial^{k_2}}{\partial y^{k_2}}u(x,y) = 2^{-k_2/2}u_\infty^{1+k_2/2}\varepsilon^{-k_2/2}x^{-k_2/2}f^{(k_2+1)}(\eta), \quad k_2\leqslant 4, \quad \eta=\eta_{(5)}(x,y;\varepsilon)\end{aligned} \qquad (25)$$

and for the function $v(x,y)$ we have representation (5). Taking into account the last representations in (25) and also the estimates for the derivatives of the function $f(\eta)$, we find

$$\begin{gathered}\left|u(x,y)\left(\frac{\partial}{\partial x}-\delta_{\bar{x}}\right)u(x,y)\right|\leqslant Mh_1 \quad (x,y)\in G_h\\ \varepsilon\left(\delta_{y\bar{y}}-\frac{\partial^2}{\partial y^2}\right)u(x,y)\geqslant mh_2^2(\varepsilon^{1/2}+h_2)^{-2}\\ m\eta^2h_2(\varepsilon^{1/2}+h_2)^{-1}\leqslant -v(x,y)\left(\delta_{\bar{y}}-\frac{\partial}{\partial y}\right)u(x,y)\leqslant M\eta^2h_2(\varepsilon^{1/2}+h_2)^{-1}\\ (x,y)\in G_h,\quad \eta\leqslant M_0,\quad \eta=\eta(x,y;\varepsilon)\end{gathered}\tag{26}$$

From estimates (26) it follows that under the condition

$$\gamma_{(1)}=\gamma_{(2)}=1\tag{27}$$

the error in the approximation of the solution of the boundary value problem is of order 1 for the terms of the equation which contain the y-derivatives, when $\eta\leqslant M_0$ and the stepsize h_2 is commensurable with $\varepsilon^{1/2}$. The error for the term involving the derivatives in x is ε-uniformly small for small values of h_1 on the whole domain $\bar{G}$.

Note that under condition (27) and for $\eta\leqslant m_0$, the main term of the truncation error is generated by errors caused by the numerical approximation of the second derivatives. These satisfy the bounds

$$\begin{aligned}48^{-1}u_\infty^3\varepsilon^{-1}h_2^2x^{-2}\min_{\eta_1}f^{(5)}(\eta_1)&\leqslant\varepsilon\left(\delta_{y\bar{y}}-\frac{\partial^2}{\partial y^2}\right)u(x,y)\\ &\leqslant 48^{-1}u_\infty^3\varepsilon^{-1}h_2^2x^{-2}\max_{\eta_2}f^{(5)}(\eta_2),\quad \eta(x,y)\leqslant m_0\end{aligned}\tag{28}$$

where $\eta_1,\eta_2\in[\eta(x,y^{j-1}),\eta(x,y^{j+1})],(x,y^j)\in G_h$.

In the variables x,ξ, where $\xi=\varepsilon^{-1/2}y$, the domain G transforms into the domain G_ξ, and the first equation from (24) takes the form

$$\begin{aligned}L^{1*0}(u^0(x,\xi))\equiv{}&\frac{\partial^2}{\partial\xi^2}u^0(x,\xi)-u^0(x,\xi)\frac{\partial}{\partial x}u^0(x,\xi)\\ &-\tilde{v}^0(x,\xi)\frac{\partial}{\partial\xi}u^0(x,\xi)=0\quad (x,\xi)\in G_\xi\end{aligned}\tag{29}$$

where $u^0(x,\xi)=u(x,y(\xi))$, $v^0(x,\xi)=v(x,y(\xi))$, $\tilde{v}^0(x,\xi)=\varepsilon^{-1/2}v^0(x,\xi)$; $u^0(x,\xi)=u_\infty f'(\eta^0)$, $\tilde{v}^0(x,\xi)=(2^{-1}u_\infty x^{-1})^{1/2}(\eta^0f'(\eta^0)-f(\eta^0))$, $\eta^0=\eta^0(x,\xi)=(2^{-1}u_\infty x^{-1})^{1/2}\xi$. The differential equation (29) in the variables x,ξ does not depend on ε. The discrete equation (22a) in these new variables takes the form

$$\begin{aligned}\Lambda^{1*0}(u^{h0}(x,\xi))\equiv{}&\gamma_{(2)}^0\delta_{\xi\bar{\xi}}u^{h0}(x,\xi)-u^{h0}(x,\xi)\delta_{\bar{x}}u^{h0}(x,\xi)\\ &-\gamma_{(1)}^0\tilde{v}^0(x,\xi)\delta_{\bar{\xi}}u^{h0}(x,\xi)=0\quad (x,\xi)\in G_{h\xi}\end{aligned}\tag{30}$$

where $u^{h0}(x,\xi)=u^h(x,y(\xi))$, $\gamma^0_{(i)}=\gamma^0_{(i)}(x,\xi;\varepsilon,h_1,h_{2\xi})=\gamma_{(i)}(x,y(\xi);\varepsilon,h_1,h_2=h_2(h_{2\xi}))$, $i=1,2$, $h_{2\xi}=\varepsilon^{-1/2}h_2$. The coefficient $\tilde{v}^0(x,\xi)$ from the grid equation (30) is independent of ε, and also the parameter ε does not influence the mesh $\bar{G}_{h\xi}$, which is defined only by its stepsizes h_1 and $h_{2\xi}$ with respect to the variables x,ξ. Because (29) and the mesh $\bar{G}_{h\xi}$ do not depend on the value of the parameter ε, it is natural to seek a numerical approximation of (29) on the mesh $\bar{G}_{h\xi}$ in form (30), based on a fitted operator method, where the coefficients, in particular, the fitting coefficients $\gamma^0_{(i)}$, are independent of the parameter ε:

$$\gamma^0_{(i)}=\gamma^0_{(i)}(x,\xi;h_1,h_{2\xi}),\quad i=1,2 \tag{31}$$

The fitting coefficients are assumed to be bounded (the monotonicity of the scheme is not used)

$$|\gamma^0_{(i)}(x,\xi;h_1,h_{2\xi})|\leqslant M\quad (x,\xi)\in G_{h\xi},\ \ i=1,2 \tag{32}$$

In this class of difference schemes we seek to construct fitted operator schemes.

Note that the largest contribution to the error of the solution of (30) is the term $\gamma^0_{(2)}\,\delta_{\xi\bar{\xi}}\,u^{h0}(x,\xi)$. The fitting coefficient $\gamma^0_{(2)}$ for fixed values of x,ξ essentially depends on the quantity $f^{(5)}(\eta^0)$ with $\eta^0=\eta^0(x,\xi;u_\infty)$, which is a non-linear function of u_∞.

Taking into account estimates (26) and (28), we establish, similarly to [8, 20], that in the case of the Prandtl problem (2), (7) and (1) there are no fitted operator schemes (22a), (22b) and (23) approximating problem (24), for which the functions $u^h(x,y)$ converge to the function $u(x,y)$ ε-uniformly.

Theorem 1

Assume that for the boundary value problem (24) we have constructed the finite difference fitted scheme (22a) and (22b) on the mesh $\bar{G}_{h(23)}$, and let the grid equations have form (22a) and (30), (31) on the meshes G_h and $G_{h\xi}$, respectively. In the class of finite difference schemes under consideration (satisfying (32)) there does not exist a scheme, whose solutions converge ε-uniformly as $h_1,h_2\to 0$.

Let us sketch the proof of Theorem 1. For more details we refer the reader to [20], where a similar statement was proved for linear problems with a parabolic boundary layer.

The proof is performed by the contradiction method. Assume that on the grid $\bar{G}_{h(23)}$ there exists a finite difference scheme which converge ε-uniformly. Let us study this scheme.

By $U_j(x,y)=(u_j(x,y),v_j(x,y))$, $j=1,2,\ldots,J$ we denote the solution of problem (4) for $u_\infty=j$. Let $u^h_j(x,y)$, $(x,y)\in\bar{G}_h$ be the solution of the difference scheme (22a), (22b) and (23) which approximates problem (24) related to the function $U_j(x,y)$. We denote $\omega_j(x,y)=u^h_j(x,y)-u_j(x,y)$, $(x,y)\in\bar{G}_h$, $j=1,\ldots,J$.

Assume $\xi^0=2h_{2\xi}$, $h_{2\xi}=m_1$. We consider equation (29) on the set

$$G^0_\xi=(x_0,x^0]\times(0,\xi^0]=\{(x,\xi):x_0<x\leqslant x^0,\ \ 0<\xi<2h_{2\xi}\}$$

This set corresponds to the set G^0 in the variables x,y. On the set $\bar{G}^0$ the grid $\bar{G}^0_h=\bar{G}^0\cap\bar{G}_h$ is defined. We shall consider the functions $\omega_j(x,y)$ for $(x,y)\in\bar{G}^0_h$. The functions $\omega_j(x,y)$ and

$\omega_j^0(x,\xi)$ are solutions of the following problems:

$$\begin{aligned}\Lambda_{(33)}(\omega_j(x,y)) &= F_{(33)}^j(x,y) \quad (x,y)\in G_h^0\\ \omega_j(x,y) &= \varphi_{(33)}^j(x,y) \quad (x,y)\in S_h^0\end{aligned} \tag{33}$$

$$\begin{aligned}\Lambda_{(34)}^0(\omega_j^0(x,\xi)) &= F_{(34)}^j(x,\xi) \quad (x,\xi)\in G_{h\xi}^0\\ \omega_j^0(x,\xi) &= \varphi_{(34)}^j(x,\xi) \quad (x,\xi)\in S_{h\xi}^0, \quad j=1,\ldots,J\end{aligned} \tag{34}$$

Here

$$\begin{aligned}\Lambda_{(33)}(\omega_j(x,y)) &\equiv \Lambda_{(22a)}^{1*}(\omega_j(x,y)) - [u_j(x,y)\delta_{\bar{x}} + \delta_{\bar{x}}u_j(x,y)]\omega_j(x,y)\\ F_{(33)}^j(x,y) &= -\Lambda_{(22a)}^{1*}(u_j(x,y)) \quad (x,y)\in G_h^0\\ \varphi_{(33)}^j(x,y) &= u_j^h(x,y) - u_j(x,y) \quad (x,y)\in S_h^0\\ \Lambda_{(34)}^0(\omega_j^0(x,\xi)) &\equiv \Lambda_{(30)}^{1*0}(\omega_j^0(x,\xi)) - [u_j^0(x,\xi)\delta_{\bar{x}} + \delta_{\bar{x}}u_j^0(x,\xi)]\omega_j^0(x,\xi)\\ F_{(34)}^j(x,\xi) &= -\Lambda_{(30)}^{1*0}(u_j^0(x,\xi)) \quad (x,\xi)\in G_{h\xi}^0\\ \varphi_{(34)}^j(x,\xi) &= u_j^{h0}(x,\xi) - u_j^0(x,\xi) \quad (x,\xi)\in S_{h\xi}^0\end{aligned}$$

the operators $\Lambda_{(33)}(\omega_j(x,y))$, $\Lambda_{(22a)}^{1*}(u_j(x,y))$ (operators $\Lambda_{(34)}^0(\omega_j^0(x,\xi))$, $\Lambda_{(30)}^{1*0}(u_j^0(x,\xi))$) contain the functions $v_j(x,y)$ (functions $\tilde{v}_j^0(x,\xi)=\varepsilon^{-1/2}v_j^0(x,\xi)$).

It is convenient to introduce auxiliary 'fitting' coefficients $\gamma_{(i)}^j$ by setting

$$\begin{aligned}\gamma_{(1)}^j(x,y;h_2,u_j(\cdot)) &= \frac{\partial}{\partial y}u_j(x,y)[\delta_{\bar{y}}\,u_j(x,y)]^{-1}\\ \gamma_{(2)}^j(x,y;h_2,u_j(\cdot)) &= \frac{\partial^2}{\partial y^2}u_j(x,y)[\delta_{y\bar{y}}u_j(x,y)]^{-1}, \quad j=1,\ldots,J\end{aligned}$$

Taking into account these coefficients, we obtain the relations

$$\begin{aligned}F_{(33)}^j(x,y) &= \varepsilon(\gamma_{(2)} - \gamma_{(2)}^j)\delta_{y\bar{y}}u_j(x,y)\\ &\quad - (\gamma_{(1)} - \gamma_{(1)}^j)v_j(x,y)\,\delta_{\bar{y}}\,u_j(x,y) + u_j(x,y)\left(\frac{\partial}{\partial x} - \delta_{\bar{x}}\right)u_j(x,y)\\ F_{(34)}^j(x,\xi) &= (\gamma_{(2)}^0 - \gamma_{(2)}^{j0})\delta_{\xi\bar{\xi}}\,u_j^0(x,\xi)\\ &\quad - (\gamma_{(1)}^0 - \gamma_{(1)}^{j\,0})\tilde{v}_j^0(x,\xi)\delta_{\bar{\xi}}u_j^0(x,\xi) + u_j^0(x,\xi)\left(\frac{\partial}{\partial x} - \delta_{\bar{x}}\right)u_j^0(x,\xi)\end{aligned}$$

where $\gamma_{(i)}^{j0}(x,\xi;h_{2\xi},u_j^0(\cdot)) = \gamma_{(i)}^j(x,\ y(\xi);h_2=h_2(h_{2\xi}),u_j(\cdot))$.

Instead of problem (33) we consider the auxiliary problem

$$\begin{aligned} \Lambda_{(33)}(\omega_j(x,y)) &= F^j_{(33)}(x,y) \quad (x,y)\in G^0_h \\ \omega_j(x,y) &= 0, \quad (x,y)\in S^0_h,\ \ j=1,\dots,J \end{aligned} \tag{35}$$

To compute the function $\omega_j(x,y)$, i.e. the solution of problem (35), we must solve the boundary value problem

$$\begin{aligned} L^{1*}_{(24)}(u(x,y)) &= 0 \quad (x,y)\in G^0 \\ u(x,y) &= u_j(x,t) \quad (x,y)\in S^0,\ \ j=1,\dots,J \end{aligned} \tag{36}$$

and the corresponding difference scheme

$$\begin{aligned} \Lambda^{1*}_{(22a)}(u^h(x,y)) &= 0 \quad (x,y)\in G^0_h \\ u^h(x,y) &= u_j(x,y) \quad (x,y)\in S^0_h,\ \ j=1,\dots,J \end{aligned} \tag{37}$$

Here $L^{1*}=L^{1*}(v_j(\cdot))$, $\Lambda^{1*}=\Lambda^{1*}(v_j(\cdot))$; $\omega_j(x,y)=u^h_j(x,y)-u_j(x,y)$, $u_j(x,y)$ and $u^h_j(x,y)$ are the solutions of problems (36) and (37).

Assuming that the solution of problem (37) converges ε-uniformly to the solution of problem (36), we come to a contradiction. By the assumption, we have

$$|\omega_j(x,y)|\leqslant \lambda_1(h_1,h_2) \quad (x,y)\in \bar{G}^0_h,\ \ j=1,\dots,J \tag{38}$$

where $\lambda_1(h_1,h_2)\to 0$ for $h_1,h_2\to 0$.

In the variables x,ξ problem (35) takes the form

$$\begin{aligned} \Lambda^0_{(34)}(\omega^0_j(x,\xi)) &= F^j_{(34)}(x,\xi) \quad (x,\xi)\in G^0_{h\xi} \\ \omega^0_j(x,\xi) &= 0 \quad (x,\xi)\in S^0_{h\xi},\ \ j=1,\dots,J \end{aligned} \tag{39}$$

Let us introduce the function $w_j(x)=\omega^0_j(x,h_{2\xi})$. This function is the solution of the discrete Cauchy problem

$$\begin{aligned} \Lambda_{(40)}w_j(x) &\equiv (1+\beta(x))\delta_{\bar{x}}w_j(x)=F_{(40)}(x), \quad x\in\omega_1 \\ w_j(x) &= 0, \quad x=x_0,\ \ j=1,\dots,J \end{aligned} \tag{40}$$

Here $\bar{\omega}_1$ is a uniform mesh on $[x_0,x^0]$ with stepsize h_1, $F_{(40)}(x)=\sum_{k=1}^{4}F_k(x)$,

$$F_1(x) = -(\gamma^0_{(1)}-\gamma^{j0}_{(1)})(u^0_j(x,\xi))^{-1}\tilde{v}^0_j(x,\xi)\delta_{\bar{\xi}}u^0_j(x,\xi)$$

$$F_2(x) = (\gamma^0_{(2)}-\gamma^{j0}_{(2)})(u^0_j(x,\xi))^{-1}\,\delta_{\xi\bar{\xi}}u^0_j(x,\xi)$$

$$F_3(x) = \left(\frac{\partial}{\partial x}-\delta_{\bar{x}}\right)u^0_j(x,\xi)$$

$$F_4(x) = -(u^0_j(x,\xi))^{-1}\omega^0_j(x,\xi)[\delta_{\bar{x}}u^0_j(x,\xi)+\gamma^0_{(1)}h^{-1}_{2\xi}\tilde{v}^0_j(x,\xi)+\gamma^0_{(2)}h^{-2}_{2\xi}]$$

$$\beta(x) = (u^0_j(x,\xi))^{-1}\omega^0_j(x,\xi), \quad \xi=h_{2\xi},\ \ j=1,\dots,J$$

Let $h_1, h_2 \to 0$. Note that the value of $h_{2\xi} = \varepsilon^{-1/2} h_2 = m_1$ is sufficiently small and bounded away from zero (by the choice of ε), and also $u_j^0(x,\xi) \geqslant m_2$ for $\xi = h_{2\xi}$, $x \in [x_0, x^0]$. By virtue of condition (38) and also because the functions $u_j^0(x,\xi)$, $\tilde{v}_j^0(x,\xi)$ are smooth with respect to x, the functions $\beta(x)$, $F_3(x)$ and $F_4(x)$ become arbitrarily small for sufficiently small values of h_1, h_2. Thus, the functions $F_1(x)$ and $F_2(x)$ are the main terms of the right-hand side of equation (40).

Let us consider the functions $F_1(x), F_2(x)$. These functions vanish for $\gamma_{(i)}^0 = \gamma_{(i)}^{j0}$, $i = 1, 2$. Note that the function $f(\eta)$, i.e. the solution of problem (6), can be decomposed (in virtue of the differential equation) as follows:

$$f(\eta) = 2^{-1} f''(0)\eta^2 [1 - 2(5!)^{-1} f''(0)\eta^3 + 22(8!)^{-1}(f''(0))^2\eta^6 + \mathcal{O}(\eta^9)]$$

Taking account of (5), we find

$$\gamma_{(1)}^{j0} = 1 - 8^{-1} f''(0) j^{3/2} \eta_1^3 + \mathcal{O}(h_{2\xi}^6)$$

$$\gamma_{(2)}^{j0} = 6 \times 7^{-1}(1 - 420^{-1} 11^2 f''(0) j^{3/2} \eta_1^3) + \mathcal{O}(h_{2\xi}^6)$$

$$(u_j^0(x,\xi))^{-1} \delta_{\xi\xi} u_j^0(x,\xi) = -7 \times 24^{-1} f''(0) x^{-1} j^{3/2} \eta_1 \times [1 - 46 \times 105^{-1} f''(0) j^{3/2} \eta_1^3 + \mathcal{O}(h_{2\xi}^6)]$$

$$(u_j^0(x,\xi))^{-1} \tilde{v}_j^0(x,\xi) \delta_{\xi} u_j^0(x,\xi) = 4^{-1} f''(0) x^{-1} j^{1/2} (\eta_1)^{-1} \times [1 - 15^{-1} f''(0) j^{3/2} \eta_1^3 + \mathcal{O}(h_{2\xi}^6)], \quad j = 1, \ldots, J$$

where $\eta_1 = (2x)^{-1/2} h_{2\xi}$. Since the coefficients $\gamma_{(1)}^0$, $\gamma_{(2)}^0$ do not depend on u_∞ (and on the value j), by choosing j we can alter the function $F_{(40)}(x)$ (and the integral, with respect to x, from this function on the segment $[x_0, x^0]$) by a quantity of the order $h_{2\xi}^{-1}$ (of the order $(x^0 - x_0) h_{2\xi}^{-1}$). Under suitable choice of the value j, the variation of the function $w_j(x)$ on the interval $[x_0, x^0]$ reaches a quantity of the order $(x^0 - x_0) h_{2\xi}^{-1}$ for $h_1, h_2 \to 0$, which contradicts condition (38). Consequently, the solution of problem (37) does not converge to the solution of problem (36) ε-uniformly.

It is easily seen that the solution of problem (22a), (22b) and (23) (where the coefficients $\gamma_{(i)(22b)}$ are independent of u_∞) do not converge ε-uniformly as well. This completes the proof of Theorem 1.

Remark 1

The statement of Theorem 1 remains valid also if condition (31) is violated, that is, the coefficients $\gamma_{(i)}^0$ depend on ε, and also condition (32) is replaced by the condition

$$|\gamma_{(i)}^0(x, \xi; h_1, h_{2\xi}, \varepsilon)| \leqslant M \quad (x,\xi) \in G_{h\xi}, \ \ i = 1, 2$$

Remark 2

By a similar way it can be shown that in the case of difference schemes on stencils with a finite number of nodes there do not exist fitted operator schemes convergent ε-uniformly, if the fitting coefficients are independent of the value u_∞.

Remark 3
But if the fitting coefficients in the difference scheme (22a), (22b) and (23) depend on u_∞, we have the following representation for these coefficients $\gamma_{(i)}$:

$$\gamma_{(1)}(x,y;\varepsilon,u_\infty,h_1,h_2)=\frac{f''(\eta)}{\delta_{\bar\eta}\,f'(\eta)},\quad \gamma_{(2)}(x,y;\varepsilon,u_\infty,h_1,h_2)=\frac{f'''(\eta)}{\delta_{\eta\bar\eta}\,f'(\eta)}$$

where $\eta=\eta_{(5)}(x,y;\varepsilon,u_\infty)$, $\delta_{\eta\bar\eta}v(\eta)=(h_\eta)^{-1}[\delta_\eta v(\eta)-\delta_{\bar\eta}v(\eta)]$, $\delta_\eta v(\eta)=(h_\eta)^{-1}[v(\eta+h_\eta)-v(\eta)]$, $h_\eta=(2^{-1}\,\varepsilon^{-1}\,u_\infty\,x^{-1})^{1/2}h_2$.

Remark 4
If we look at the difference schemes in question, namely, the classical scheme (22a), (28) and the fitted scheme (22a) and (22b) on the mesh (23), their fitting coefficients $\gamma_{(i)}$ determined by (28) and (22b) do not depend on the value of u_∞, which defines the solution of problem (24). We call these coefficients $\gamma_{(i)}$ the generalized fitting coefficients. It follows from the above considerations that in order to construct ε-uniformly convergent schemes (both truly fitted operator schemes and schemes consisting of a standard difference operator) whose generalized fitting coefficients are independent of u_∞, the use of *piecewise-uniform* meshes condensing in the parabolic boundary layer region is necessary. A similar conclusion is valid also in the case of the Prandtl problem (2), (1), for which (24) is a model problem.

7. NUMERICAL EXPERIMENTS FOR THE PRANDTL PROBLEM

It is well known that, in the case of linear problems with parabolic boundary layers, standard finite difference schemes on piecewise-uniform fitted meshes yield numerical solutions that converge ε-uniformly (for the convergence proof and corroborant numerical results we refer the reader, e.g. to References [8, 9, 16, 15]).

In the case of the non-linear problem (2), (1) and (7) having the self-similar solution, the efficiency of scheme (12) and (13) (scheme (14) and (13)) is demonstrated (see the error Tables I–IV) by comparing the numerical solutions to the *reference* solution generated from the computed solution to Blasius' problem (6). The Blasius problem was numerically solved by scheme (18a), (18b) and (16) on a sufficiently fine mesh, namely with the number of mesh intervals $N=8192$, which provided the required accuracy in the reference solution (5). The parameters T,R of scheme (18a) and (18b) and the mesh size h are defined by (19) and (21), respectively, where $M_1=1$, $M_2=8$ were taken for all values of ε.

Graphs of the numerical approximations $U^h(x,y)=(u^h(x,y),v^h(x,y))$ to the velocity components $U(x,y)=(u(x,y),v(x,y))$ with $N=32$ are shown in Figures 1 and 2 for $\varepsilon=0.01$ and ε=0.00001, respectively. We see from these graphs that the velocity components $u(x,y),v(x,y)$ contain the boundary layer in a neighbourhood of the boundary at $y=0$, moreover, the second component $v(x,y)$ unboundedly grows as $x\to 0$.

The computed maximum pointwise errors for the normalized velocity components $u(x,y)$ and $\varepsilon^{-1/2}v(x,y)$ are given in Tables I and II, respectively, for various values of ε and N. Tables III and IV list the computed maximum pointwise errors for the normalized partial derivatives $(\partial/\partial x)u(x,y)$ and $\varepsilon^{1/2}(\partial/\partial y)u(x,y)$, respectively.

We observe from the above error tables that the maximum nodal errors decrease as N increases for each value of the parameter ε and that the maximum global error for a

Table I. Computed maximum nodal errors for $u(x, y)$ for various values of ε and N.

ε	8	16	32	64	128	256	512
2^0	0.420D − 2	0.459D − 2	0.287D − 2	0.166D − 2	0.898D − 3	0.450D − 3	0.211D − 3
2^{-2}	0.509D − 1	0.248D − 1	0.124D − 1	0.622D − 2	0.312D − 2	0.157D − 2	0.792D − 3
2^{-4}	0.207D + 0	0.787D − 1	0.352D − 1	0.167D − 1	0.817D − 2	0.404D − 2	0.202D − 2
2^{-6}	0.220D + 0	0.115D + 0	0.616D − 1	0.326D − 1	0.156D − 1	0.762D − 2	0.378D − 2
2^{-8}	0.213D + 0	0.114D + 0	0.616D − 1	0.340D − 1	0.189D − 1	0.105D − 1	0.581D − 2
2^{-10}	0.211D + 0	0.114D + 0	0.616D − 1	0.340D − 1	0.189D − 1	0.105D − 1	0.581D − 2
.	.	.	.	.	.	.	.
.	.	.	.	.	.	.	.
.	.	.	.	.	.	.	.
2^{-20}	0.208D + 0	0.113D + 0	0.616D − 1	0.340D − 1	0.189D − 1	0.105D − 1	0.581D − 2

Table II. Computed maximum nodal errors for $\varepsilon^{-1/2}v(x, y)$ for various values of ε and N.

ε	8	16	32	64	128	256	512
2^0	0.533D + 0	0.374D + 0	0.213D + 0	0.108D + 0	0.536D − 1	0.268D − 1	0.142D − 1
2^{-2}	0.106D + 1	0.677D + 0	0.371D + 0	0.194D + 0	0.101D + 0	0.531D − 1	0.287D − 1
2^{-4}	0.396D + 1	0.163D + 1	0.763D + 0	0.382D + 0	0.197D + 0	0.104D + 0	0.562D − 1
2^{-6}	0.457D + 1	0.271D + 1	0.154D + 1	0.849D + 0	0.416D + 0	0.215D + 0	0.114D + 0
2^{-8}	0.448D + 1	0.269D + 1	0.154D + 1	0.893D + 0	0.523D + 0	0.309D + 0	0.183D + 0
2^{-10}	0.437D + 1	0.268D + 1	0.154D + 1	0.893D + 0	0.523D + 0	0.309D + 0	0.183D + 0
.	.	.	.	.	.	.	.
.	.	.	.	.	.	.	.
.	.	.	.	.	.	.	.
2^{-20}	0.424D + 1	0.267D + 1	0.154D + 1	0.893D + 0	0.523D + 0	0.309D + 0	0.183D + 0

Table III. Computed maximum nodal errors for $(\partial/\partial x)u(x, y)$ for various values of ε and N.

ε	8	16	32	64	128	256	512
2^0	0.614D + 0	0.444D + 0	0.256D + 0	0.130D + 0	0.668D − 1	0.344D − 1	0.178D − 1
2^{-2}	0.900D + 0	0.633D + 0	0.372D + 0	0.201D + 0	0.105D + 0	0.556D − 1	0.307D − 1
2^{-4}	0.189D + 1	0.114D + 1	0.650D + 0	0.360D + 0	0.194D + 0	0.105D + 0	0.573D − 1
2^{-6}	0.198D + 1	0.167D + 1	0.121D + 1	0.759D + 0	0.397D + 0	0.210D + 0	0.113D + 0
2^{-8}	0.197D + 1	0.167D + 1	0.121D + 1	0.798D + 0	0.496D + 0	0.300D + 0	0.180D + 0
2^{-10}	0.196D + 1	0.167D + 1	0.121D + 1	0.798D + 0	0.496D + 0	0.300D + 0	0.180D + 0
.	.	.	.	.	.	.	.
.	.	.	.	.	.	.	.
.	.	.	.	.	.	.	.
2^{-20}	0.195D + 1	0.167D + 1	0.121D + 1	0.798D + 0	0.496D + 0	0.300D + 0	0.180D + 0

Table IV. Computed maximum nodal errors for $\varepsilon^{1/2}(\partial/\partial y)u(x, y)$ for various values of ε and N.

ε	8	16	32	64	128	256	512
2^{0}	0.703D − 1	0.357D − 1	0.180D − 1	0.914D − 2	0.471D − 2	0.249D − 2	0.139D − 2
2^{-2}	0.193D + 0	0.111D + 0	0.603D − 1	0.315D − 1	0.162D − 1	0.819D − 2	0.414D − 2
2^{-4}	0.266D + 0	0.140D + 0	0.703D − 1	0.357D − 1	0.180D − 1	0.914D − 2	0.471D − 2
2^{-6}	0.279D + 0	0.192D + 0	0.118D + 0	0.703D − 1	0.357D − 1	0.180D − 1	0.914D − 2
2^{-8}	0.279D + 0	0.192D + 0	0.118D + 0	0.733D − 1	0.432D − 1	0.248D − 1	0.141D − 1
.	.	.	.	.	.	.	.
.	.	.	.	.	.	.	.
.	.	.	.	.	.	.	.
2^{-20}	0.279D + 0	0.192D + 0	0.118D + 0	0.733D − 1	0.432D − 1	0.248D − 1	0.141D − 1

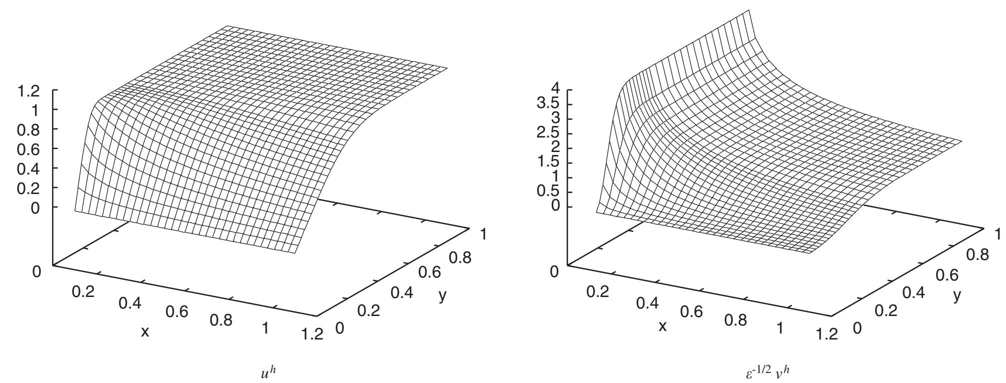

Figure 1. Graph of $U^h(x, y)$ for $\varepsilon = 0.01$ and $N = 32$.

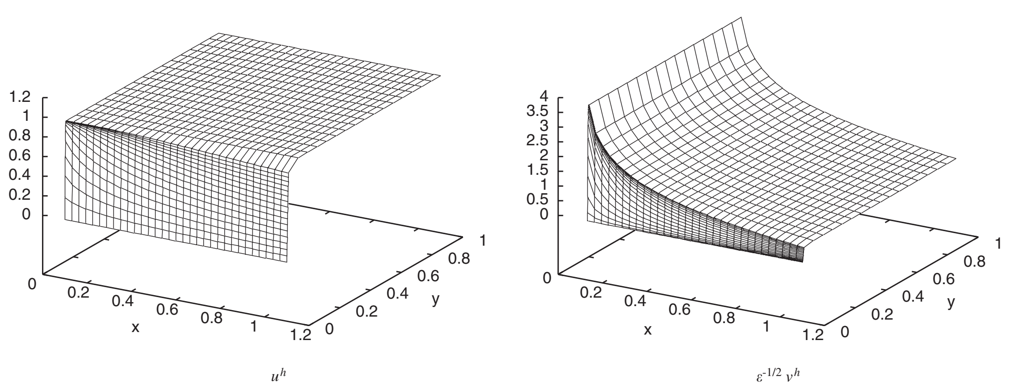

Figure 2. Graph of $U^h(x, y)$ for $\varepsilon = 0.00001$ and $N = 32$.

particular N and all available values of ε, the ε-uniform error, also decreases with increasing N. These results show experimentally that the difference scheme (12a)–(12d) and (13) allows us to approximate ε-uniformly (with controllable accuracy for arbitrary values of the Reynolds number) the functions $u(x,t)$, $\varepsilon^{-1/2}v(x,y)$ and the partial derivatives $(\partial/\partial x)u(x,y)$, $\varepsilon^{1/2}(\partial/\partial y)u(x,y)$; the order of ε-uniform convergence is close to 1. Analogous results are obtained for the numerical approximations to the partial derivatives $\varepsilon^{-1/2}(\partial/\partial x)v(x,y)$, and $(\partial/\partial y)v(x,y)$.

8. CONCLUSION

In this paper, numerical approximations to the solution of Prandtl's boundary value problem for the boundary layer equations on a flat plate are given in a region including the boundary layer, but outside a neighbourhood of its leading edge. A finite difference scheme based on a monotone finite difference operator and piecewise-uniform meshes, which are refined in the vicinity of the parabolic boundary layer, is constructed. The Blasius problem is numerically solved to obtain a self-similar solution and this was used as a reference solution to determine the accuracy of the numerical approximations. It is shown that the numerical approximations converge independently of the Reynolds number. Thus, the direct numerical method based on the fitted mesh method suggested in the paper allows us to approximate both the components of the solution and their derivatives with controllable ε-uniform accuracy; $\varepsilon = Re^{-1}$.

The applicability of fitted operator methods for Prandtl's problem is also discussed. As is shown, the technique based on fitted operator methods does not allow us to obtain ε-uniform numerical approximations for flow problems with a parabolic boundary layer, in particular, for Prandtl's problem for flow past a plate. Thus, the use of meshes refined in the parabolic layer region is necessary to construct ε-uniform direct numerical methods for flow problems.

The numerical technique presented in the paper may be also applicable to the construction and study of ε-uniform direct numerical methods for more complicated problems of flow past a wedge or a body of revolution, flow in converging channels, stagnation flow and others.

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